3. RG and the 1D Ising Model with an external magnetic field:Transfer matrix method.

We are looking to find expressions for K', H', and f(H,K) that satisfy the relationship

$$\underline{\mathbf{T}}' = \underline{\mathbf{T}} \cdot \underline{\mathbf{T}}$$

$$f(K,H)\begin{pmatrix} e^{K'+H'} & e^{-K'} \\ e^{-K'} & e^{K'-H'} \end{pmatrix} = \begin{pmatrix} e^{2K+2H} + e^{-2K} & e^{-H} + e^{H} \\ e^{-H} + e^{H} & e^{2K-2H} + e^{-2K} \end{pmatrix}$$

Evidently there are three unique matrix elements and three unknown constants, forming a system of equations with a single unique solution:

$$e^{2K+2H} + e^{-2K} = 2e^{H}\cosh(2K+H) = f(K,H)e^{K'+H'} \tag{1}$$

$$e^{2K-2H} + e^{-2K} = 2e^{-H}\cosh(2K-H) = f(K,H)e^{K'-H'} \tag{2}$$

$$e^{-H} + e^{H} = 2\cosh(H) = f(K,H)e^{-K'} \tag{3}$$

$$\frac{(1)(2)}{(3)^2} = \frac{\cosh(2K+H)\cosh(2K-H)}{\cosh^2(H)} = e^{4K'}$$

$$\boxed{K' = \frac{1}{4}\ln\left(\frac{\cosh(2K+H)\cosh(2K-H)}{\cosh^2(H)}\right)} \tag{4}$$

$$\frac{(1)}{(2)} = e^{2H}\frac{\cosh(2K+H)}{\cosh(2K-H)} = e^{2H'}$$

$$\boxed{H' = H + \frac{1}{2}\ln\left(\frac{\cosh(2K+H)}{\cosh(2K-H)}\right)} \tag{5}$$

$$(1)(2)(3)^2 = f^4 = 16\cosh(2K+H)\cosh(2K-H)\cosh^2(H)$$

$$\boxed{\begin{aligned} f(H,K) &= 2\left(\cosh(2K+H)\cosh(2K-H)\cosh^2(H)\right)^{1/4} \\ \ln f &= \frac{1}{4}\left(\cosh(2K+H)\cosh(2K-H)\cosh^2(H)\right) \end{aligned}} \tag{6}$$

Equations (4), (5), and (6) are the RG recursion relationships for the 1D Ising model with nonzero H. Note that they reduce to the equations in Chandler 5.6 if H = 0.

To find g(H',K'), start with the equation for Z_N:

$$Z_N = \text{trace}\left(\underline{\mathbf{T}}^N\right) = \text{trace}\left(\underline{\mathbf{T}}'^{N/2}\right)$$

$$= \text{trace}\left(\left(f(K,H)\begin{pmatrix} e^{K'+H'} & e^{-K'} \\ e^{-K'} & e^{K'-H'} \end{pmatrix}\right)^{N/2}\right) = \text{trace}\left(f(K,H)^{N/2}\begin{pmatrix} e^{K'+H'} & e^{-K'} \\ e^{-K'} & e^{K'-H'} \end{pmatrix}^{N/2}\right)$$

$$= f(K,H)^{N/2}\,\text{trace}\left(\begin{pmatrix} e^{K'+H'} & e^{-K'} \\ e^{-K'} & e^{K'-H'} \end{pmatrix}^{N/2}\right)$$

$$= f(K,H)^{N/2}\,\text{trace}\left(\mathbf{T}_{\frac{N}{2}}^{\;N/2}\right); \mathbf{T}_{\frac{N}{2}} \equiv \left\{\text{ Transfer Matrix for an Ising Lattice with } N/2 \text{ spins}\right\}$$

$$\text{so } Z_N = f(K,H)^{N/2}\, Z_{N/2}$$

and

$$\ln(Z_N) = Ng(H,K) = \frac{N}{2}\ln\left(f(H,K)\right) + \ln Z_{N/2} = \frac{N}{2}\ln\left(f(H,K)\right) + \frac{N}{2}g(H',K')$$

$$g(H',K') = 2g(H,K) - \ln\left(f(H,K)\right)$$

$$= 2g(H,K) - \frac{1}{4}\left(\cosh(2K+H)\cosh(2K-H)\cosh^2(H)\right)$$

Fixed points occur anywhere that K and H map to themselves under another recursion cycle. These correspond to (K=0,H) and (K=∞, H=0).