HW #7, due in class on 03/03/2005

Problem 1. Infinite range Ising model

Consider the following Ising model Hamiltonian

$$H_N\{S\} = -H\sum_i S_i - \frac{J_0}{2}\sum_i\sum_j S_iS_j \tag{1}$$

Here H is the usual external field. Indeces of summation i and j run from 1 to N, where N is the number of spins in the system. Notice that there is no nearest neighbor restriction in the second double sum, in contrast to the case considered in class. This means that every spin interacts with every other spin with the equal strength given by the coupling parameter J_0. You will solve this problem using the *Hubbard-Stratonovich transformation*, a.k.a. method of auxiliary fields.

(a) Consider the total energy of a configuration of this system when all the spins point up ($S_i = +1$) and show that this energy is extensive (proportional to N) only when $J_0 = J/N$, where J is some constant. In the following we are going to use this definition of J_0.

(b) Notice that since there is no restriction on the double sum it can be written as the product of the single sums:

$$\sum_i\sum_j S_iS_j = (\sum_i S_i)(\sum_j S_j) = (\sum_i S_i)^2 \tag{2}$$

Here the second identity comes from the fact that i and j are dummy indeces of summation. Using the identity (this is the Hubbard-Stratonovich transformation)

$$e^{\frac{ax^2}{2N}} = \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi/Na}} e^{-\frac{Nay^2}{2}+axy}, \tag{3}$$

where $a > 0$, show that the partition function of this model can be written as

$$Z_N = \int_{-\infty}^{+\infty} \frac{dy}{\sqrt{2\pi/N\beta J}} e^{-Ng(y)} \tag{4}$$

where

$$g(y) = \frac{\beta J}{2}y^2 - \ln[2\cosh(\beta H + \beta J y)] \tag{5}$$

[Hint: you should identify x in this transformation with $\sum_i S_i$. After performing the transformation you still need to sum over all possible spin states. To do this exchange the order of this summation and the integration over y. Notice that now you can sum over states of every spin independently. This is the main idea of the Hubbard-Stratonovich "trick".]

(c) In the thermodynamic limit ($N \to \infty$) the integral in Eq. (4) can be evaluated exactly by the *method of steepest descent*: the integral is dominated by a *global minimum* of $g(y)$ as a function of y. That is

$$Z_N(\beta, H, J) = e^{-Ng(y_0)}, \qquad N \to \infty, \tag{6}$$

where $y_0(\beta, H, J)$ is the position of the global minimum of $g(y)$. Hence show that in this limit the magnetization $M =< S_i >$ is given by

$$M \equiv \lim_{N\to\infty} \frac{1}{\beta N} \frac{\partial \ln Z_N}{\partial H} = y_0 \tag{7}$$

(d) Consider the case of zero magnetic field $H = 0$. Solve the equation $\partial g(y)/\partial y = 0$ for y_0 graphically and show that there is a phase transition and find the transition temperature T_c. [Hint: by differentiating Eq. (5) and equating it to zero, show that the minimization can be written as a problem of finding roots of $\beta J y = f(y, \beta J)$, where the left and right hand sides come from derivatives of the first and second terms of Eq. (5) respectively and $f()$ is a function you need to determine. Plot both sides of this equation vs. y for a couple of values of βJ and show that for large βJ there is only one solution, where as for small enough βJ there are three solutions. The value of βJ at which this change occurs corresponds to the transition temperature. Calculate this transition temperature T_c in terms of the coupling parameter J. Discuss which solutions correspond to the minima and/or maxima of $g(y)$. Remember, the solution you need has to be the global minimum!]

(e) Calculate the isothermal susceptibility

$$\chi_T \equiv \frac{\partial M}{\partial H}. \tag{8}$$

For $H = 0$, show that χ_T diverges to infinity both above and below the transition temperature T_c. Find the leading singular behavior of χ_T in terms of the reduced temperature $t \equiv (T - T_c)/T_c$. [Hint: you will need to calculate the derivative of y_0 with respect to the field H. The problem is that you only have an implicit dependence of y_0 on H. Differentiate both sides of the equation that y_0 has to satisfy and solve it for $\partial y_0/\partial H$ in terms of y_0, β, J and H. Now set $H = 0$. Close to the transition temperature y_0 is small (it is actually strictly zero above the transition temperature), so retain only the leading terms in y_0. Alternatively, you can expand the equation for y_0 close to the transition from the start and derive the same result. Finally, make the change from T to the reduced temperature t and find the leading divergence. Notice, that the reduced temperature t vanishes at the transition point by definition. Use this fact to perform the leading order expansions.]

Problem 2. Mean Field Theory

In this problem you will derive thermodynamic properties of the mean field solution obtained in class.

(a) Using simple algebra one can invert the mean field equation for the magnetization $M(H)$

$$M = \tanh(H/k_B T + zJM/k_B T) \tag{9}$$

to obtain the equation for the external field $H(M)$

$$\tanh(H/k_B T) = \frac{M - \tanh(\tau M)}{1 + M \tanh(\tau M)}, \tag{10}$$

where to simplify notation we introduced $\tau \equiv T_c/T = zJ/k_B T$.

Expand both sides of this equation near the phase transition ($T = T_c$) where H and M are small. Keep only the linear term in H expansion on the LHS and terms upto the third order in M expansion of the RHS. You should obtain result of the form

$$\frac{H}{k_B T} \approx a(\tau) M + b(\tau) M^3, \tag{11}$$

where the functions $a(\tau)$ and $b(\tau)$ you need to find.

(b) Set $H = 0$ and find $M(T)$ dependence near the phase transition. Plot $M(T)$ above and below the phase transition temperature. Show that it has a power law divergence as a function of $(T - T_c)$ as T approaches the transition temperature from below. This should give you the critical exponent β, which is defined as $M \propto (T_c - T)^\beta$ as $T \to T_c^-$.

(c) Consider the dependence $H(M)$ along the critical isotherm $T = T_c$ (or in our notation $\tau = 1$). What is the critical exponent δ, which is formally defined as $H \propto |M|^\delta$ for $M \to 0$?

(d) Calculate the isothermal magnetic susceptibility χ_T by differentiating Eq. (11). Determine divergence of χ_T as the temperature approaches T_c both from above and below. (Be careful: even though $M(T)$ vanishes identically for $T > T_c$, it is not zero for $T < T_c$. Determine the the critical exponents γ and γ', which are defined as

$$\chi_T(T) \propto (T - T_c)^{-\gamma}, \quad T \to T_c^- \tag{12}$$

$$\chi_T(T) \propto (T_c - T)^{-\gamma'}, \quad T \to T_c^+ \tag{13}$$